Review of Non-Commutative Geometric Methods Applied to Particle Physics

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Abstract

This is a brief review where some basic elements of non-commutative geometry are given. The rules and ingredients that enter in the construction of the standard model and grand unification models in non-commutative geometry are summarized. A connection between some space-time supersymmetric theories and non-commutative geometry is made. The advantages and prolems of this direction are discussed.

1. Introduction

There is a continuous search for a consistent unified theory of all fundamental interactions. The main difficulty lies in unifying space-time geometry with quantum theory, and this is intrinsically linked to understanding physics at the planck scale. Since it is not possible to investigate physics directly at such high scales, we are mainly guided by considerations of mathematical consistency, elegance and simplicity. A central assumption in our forumulation of quantum dynamics is the manifold structure of space-time where the methods of commutative differential geometry apply. This assumption is not made in string theory where the space-time manifold arises in the limit when loops, in the loop space, shrink to points at lower energies. Another idea in this direction is the study of non-commutative spaces as initiated by Connes [1]. At present the list of non-commutative spaces that has been studied is limited, and only a handful of such spaces correspond to physical theories [2-6]. What is encouraging is the fact that the simplest non-commutative space taken to be the product of a continuous four-dimensional Minkowski space times a discrete set of two points, gives the standard model with the Higgs fields and gauge fields unified [2-5]. There are also other examples where the discrete set consists of more than two

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points and where the resultant models are grand unified models [6]. We shall see that this construction has many attractive features, but it will also be clear that we are still far from a final unification picture.

The aim of this review is to give the basic ingredients that go into the construction of a model based on a non-commutative space, and to spell out clearly the assumptions made, and whether improvements are possible. The emphasis here will be on giving an honest and coherent picture of where things stand and the important problems to be solved in order to make further progress.

This review is organized as follows. In section two, we give a summary of the basic elements of non-commutative geometry. In section three the steps for constructing the standard model using non-commutative geometry are listed. In section four, this is generalized to grand unified models. In section five we give a connection between some supersymmetric theories and non-commutative geometry. In section six the question of gravity in non-commutative spaces is discussed, and section seven is the conclusion.

2. Basic notions of non-commutative geometry

A smooth manifold, M, can be studied by analyzing the commutative algebra, $C^\infty(M)$, of smooth functions on M. In fact, M can be reconstructed from the structure of $C^\infty(M)$. The basic idea in non-commutative geometry [1] is to define a notion of non-commutative space in terms of a non-commutative (non-abelian) algebra, $\mathcal{A}$, which is assumed to be an <u>involutive</u> algebra. This means that there is an antilinear operation * taking $a \in \mathcal{A}$ to $a^* \in \mathcal{A}$, (a^* is the adjoint of a with $(a \cdot b)^* = b^* \cdot a^*$). We also assume that $\mathcal{A}$ contains an identity element 1. In this case one says that $\mathcal{A}$ is a unital, involutive algebra. It defines a notion of a <u>compact</u>, non-commutative space.

Given a unital, involutive algebra $\mathcal{A}$, one can define an algebra

$$\Omega^*(\mathcal{A}) \;=\; \overset{\infty}{\underset{n=0}{\oplus}} \; \Omega^n(\mathcal{A})$$

as the "universal, differential algebra" over $\mathcal{A}$, as follows: One sets $\Omega^0(\mathcal{A}) = \mathcal{A}$, and defines $\Omega^n(\mathcal{A})$ to be the linear space given by

$$\Omega^n(\mathcal{A}) = \left\{ \sum_i a_0^i \, da_1^i \ldots da_n^i \; : a_j^i \, \epsilon \mathcal{A}, \forall i, j \right\}, \qquad n = 1, 2, \cdots.$$

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Here da denotes an equivalence class of $a \in \mathcal{A}$, modulo the following relations:

$$d(a \cdot b) = (da) \cdot b + a \cdot db, \quad d1 = 0, \quad d^2 = 0.$$

An element of $\Omega^n(\mathcal{A})$ is called a <u>form of degree</u> n. Let $\alpha \in \Omega^n(\mathcal{A})$ and $\beta \in \Omega^m(\mathcal{A})$. Then one can define the product, $\alpha \cdot \beta$, of α with β, and one verifies that $\alpha \cdot \beta \in \Omega^{n+m}(\mathcal{A})$, i.e. $\alpha \cdot \beta$ is a form of degree $n + m$. With this definition of a product of forms, $\Omega^*(\mathcal{A})$ becomes an algebra. Defining

$$(da)^* = -d(a^*)$$

one immediately deduces from the definition of $\Omega^n(\mathcal{A})$ and from Leibniz rule that, for $\alpha \in \Omega^n(\mathcal{A})$, α^* is defined and is again an element of $\Omega^n(\mathcal{A})$.

One-forms play a special role as components of connections on a "line bundle" whose space of sections is given by the algebra $\mathcal{A}$. A 1-form $\rho \in \Omega^1(\mathcal{A})$ can be expressed as

$$\rho = \sum_i a^i db^i,$$

a^i, b^i in $\mathcal{A}$, and, since $d1 = 0$, we may impose the condition that

$$\sum_i a^i b^i = 1,$$

without loss of generality.

Next, we introduce the notion of a (Dirac) K-cycle for $\mathcal{A}$. Let h be a separable Hilbert space, and let D be a selfadjoint operator on h. We say that (h, D) is a (Dirac) K-cycle for $\mathcal{A}$ iff there exists an involutive representation, π, of $\mathcal{A}$ on h, i.e., a representation (or antirepresentation) of $\mathcal{A}$ satisfying $\pi(a^*) = \pi(a)^*$, with the properties that

(i) $\pi(a)$ and $[D, \pi(a)]$ are bounded operators on h, for all $a \in \mathcal{A}$; and

(ii) $(D^2 + 1)^{-1}$ is a compact operator on h. A K-cycle (h, D) for $\mathcal{A}$ is said to be (d, ∞)-summable iff the trace of $(D^2 + 1)^{-p/2}$ exists and is finite, for all $p > d$. A K-cycle (h, D) for $\mathcal{A}$ is said to be <u>even</u> iff there exists a unitary involution Γ on h, i.e. a bounded operator on h with $\Gamma^* = \Gamma^{-1} = \Gamma$, such that $[\Gamma, \pi(a)] = 0$ for all $a \in \mathcal{A}$, and $\{\Gamma, D\} = \Gamma D + D\Gamma = 0$. Otherwise (h, D) is called <u>odd</u>.

Given a K-cycle (h, D) for $\mathcal{A}$, we define a representation, π of $\Omega^*(\mathcal{A})$ on h by setting

$$\pi\left(\sum_i a_0^i da_1^i, \cdots da_n^i\right)$$

$$= \sum_i \pi(a_0^i)\left[D, \pi(a_1^i)\right] \cdots \left[D, \pi(a_n^i)\right],$$

for any element $\sum_i a_0^i da_1^i \cdots da_n^i \in \Omega^n(\mathcal{A})$, $n = 0, 1, 2, \cdots$. We also define the spaces of <u>auxiliary fields</u>

$$\mathrm{Aux} = \ker \pi + d \ker \pi,$$

where

$$\ker \pi = \overset{\infty}{\underset{n=0}{\oplus}} \left\{\sum_i a_0^i \, da_1^i \cdots da_n^i \; : \right.$$

$$\left. \pi\left(\sum_i a_0^i \, da_1^i \cdots da_n^i\right) = 0\right\},$$

and

$$d \ker \pi = \overset{\infty}{\underset{n=0}{\oplus}} \left\{\sum_i da_0^i \, da_1^i \cdots da_n^i \; : \right.$$

$$\left. \pi\left(\sum_i a_0^i \, da_1^i \cdots da_n^i\right) = 0\right\}.$$

It follows from Leibniz rule that Aux is a two-sided ideal in $\Omega^*(\mathcal{A})$, and hence $\Omega_D^*(\mathcal{A}) = \otimes^*(\mathcal{A})$ modulo Aux is a universal differential algebra. If $\sum_i a_0^i \, da_1^i \cdots da_n^i \in \Omega^n(\mathcal{A})$ then

$$\left\{\sum_i \pi(a_0^i)\left[D, \pi(a_1^i)\right] \cdots \left[D, \pi(a_n^i)\right] + \pi(\alpha) : \quad \alpha \in \mathrm{Aux}\right\}$$

represents an n-form,

$$\alpha = \sum_i a_0^i \, a_1^i \cdots da_n^i \quad \mathrm{mod \; Aux},$$

in $\Omega_D^n(\mathcal{A})$ as an equivalence class of bounded operators on the Hilbert space h.

We define the integral of a form $\alpha \in \Omega^*(\mathcal{A})$ over a non-commutative space $\mathcal{A}$ by setting

$$\int \alpha = \mathrm{Tr}_w\left(\pi(\alpha)D^{-d}\right)$$

where Tr_w is the Dixmier trace [1]. Alternatively this integral can be defined by the heat-kernel expression

$$\lim_{\epsilon \to 0} \frac{\mathrm{tr}(\pi(\alpha)e^{-\epsilon|D|^2})}{\mathrm{tr}(e^{-\epsilon|D|^2})}$$

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For more details on these somewhat abstract mathematical notions the reader is referred to [1,4,7].

3. The standard model in non-commutative geometry

One of the simplest extensions for the structure of space-time is to take it to be a product of a continuous four-dimensional manifold times a discrete set of two points [2-5]. The algebra is $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ acting on the Hilbert space $h = h_1 \otimes h_2$, where $\mathcal{A}_1 = C^\infty(M)$, and $\mathcal{A}_2 = M_2(C) \oplus M_1(C)$ the algebras of 2×2 and 1×1 matrices. The Hilbert space is that of spinors of the form $L = \begin{pmatrix} l \\ e \end{pmatrix}$ where l is a doublet and e is a singlet. The spinor L satisfies the chirality condition $\gamma_5 \otimes \Gamma_1 L = L$, where $\Gamma_1 = \mathrm{diag}(1_2, -1)$ is the grading operator. This implies that $l = l_L$ is left-handed and $e = e_R$ is right-handed, and so we can write $l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$. The Dirac operator is $D = D_1 \otimes 1 + \Gamma_1 \otimes D_2$, where $D_1 = \not\partial$ is the Dirac operator on $\mathcal{A}_1$ and D_2 is the Dirac operator on $\mathcal{A}_2$, so that

$$D_l = \begin{pmatrix} \not\partial \otimes 1_2 & \gamma_5 M_{12} \otimes k \\ \gamma_5 M_{21} \otimes k^* & \not\partial \end{pmatrix},$$

where $M_{21} = M_{12}^*$ and k is a family mixing matrix. The geometry is that of a four-dimensional manifold M times a discrete space of two points. The column M_{12} in D, the vev of the Higgs field, is taken to be $M_{12} = \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv H_0$. The elements $a \in \mathcal{A}$ have the representation $a \to \mathrm{diag}(a_1, a_2)$ where a_1 and a_2 are 2×2 and 1×1 unitary matrix-valued functions, respectively. The self-adjoint one-form ρ has the representation

$$\pi_l(\rho) = \begin{pmatrix} A_1 \otimes 1_3 & \gamma_5 H \otimes k \\ \gamma_5 H^* \otimes k^* & A_2 \otimes 1_3 \end{pmatrix},$$

where $A_1 = \sum_i a_1^i \not\partial b_1^i$, $A_2 = \sum_i a_2^i \not\partial b_2^i$ and $H = H_0 + \sum_i a_1^i H_0 b_2^i$. To be realistic, the quarks and the SU(3) gauge group must be introduced. This can be achieved by taking a bimodule structure relating two algebras $\mathcal{A}$ and $\mathcal{B}$ [2], where the algebra $\mathcal{B}$ is taken to be $M_1(C) \oplus M_3(C)$, commuting with the action of $\mathcal{A}$, and the mass matrices in the Dirac operator are taken to be zero when acting on elements of $\mathcal{B}$. Then the one-form η in $\Omega^1(\mathcal{B})$ has the simple form $\pi_l(\eta) = B_1 \mathrm{diag}(1_2, 1)$, where B_1 is a $U(1)$ gauge field associated with $M_1(C)$. The quark Hilbert space is that of the spinor $Q = \begin{pmatrix} u_L \\ d_L \\ d_R \\ u_R \end{pmatrix}$. The representation of $a \in \mathcal{A}$ is: $a \to \mathrm{diag}(a_1, a_2, \overline{a_2})$ where a_1

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is a 2×2 matrix-valued function and a_2 is a complex-valued function. The Dirac operator acting on the quark Hilbert space is

$$
D_q = \begin{pmatrix} \gamma^\mu(\partial_\mu + \ldots) \otimes 1_2 \otimes 1_3 & \gamma_5 \otimes M_{12} \otimes k' & \gamma_5 \otimes \tilde{M}_{12} \otimes k'' \\ \gamma_5 \otimes M_{12}^* \otimes k'^* & \gamma^\mu(\partial_\mu + \ldots) \otimes 1_3 & 0 \\ \gamma_5 \otimes \tilde{M}_{12}^* \otimes k''^* & 0 & \gamma^\mu(\partial_\mu + \ldots) \otimes 1_3 \end{pmatrix} \otimes 1_3,
$$

where k' and k'' are 3×3 family mixing matrices, and $\tilde{M}_{12} = \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then the one-form in $\Omega^1(\mathcal{A})$ has the representation

$$
\pi_q(\rho) = \begin{pmatrix} A_1 \otimes 1_3 & \gamma_5 H \otimes k' & \gamma_5 \tilde{H} \otimes k'' \\ \gamma_5 H^* \otimes k'^* & A_2 \otimes 1_3 & 0 \\ \gamma_5 \tilde{H}^* \otimes k''^* & 0 & \overline{A_2} \otimes 1_3 \end{pmatrix},
$$

where $\tilde{H}_a = \epsilon_{ab} H^b$. On the algebra $\mathcal{B}$ the Dirac operator has zero mass matrices, and the one form η in $\Omega^1(\mathcal{B})$ has the representation $\pi_q(\eta) = B_2 \mathrm{diag}(1_2, 1, 1)$ where B_2 is the gauge field associated with $M_3(C)$. Imposing the unimodularity condition on the algebras $\mathcal{A}$ and $\mathcal{B}$ relates the U(1) factors in both algebras [3]: $\mathrm{tr}(A_1) = 0$, $A_2 = B_1 = -\mathrm{tr}B_2 = \frac{i}{2}g_1 B$. We can then write

$$
A_1 = -\frac{i}{2}g_2 A^a \sigma_a
$$

$$
B_2 = -\frac{i}{6}g_1 B - \frac{i}{2}g_3 V^i \lambda_i
$$

where g_3 is the SU(3) guge coupling constant, and σ^a and λ^i are the Pauli and Gell-Mann matrices, respectively. It is tempting to conjecture that the unimodularity condition is related to the absence of chiral anomalies. The fermionic action for the leptons is

$$
< L, (D + \rho + \eta)L > = \int d^4x \sqrt{g}\Big(\overline{L}(D_l + \pi_l(\rho) + \pi_l(\eta))L\Big),
$$

and, for the quarks it is

$$
< Q, (D + \rho + \eta)Q > = \int d^4x \sqrt{g}\Big(\overline{Q}(D_q + \pi_q(\rho) + \pi_q(\eta))Q\Big),
$$

and these can be easily checked to reproduce the standard model lepton and quark interactions with the correct hypercharge assignments.

The bosonic action is the sum of the squares of the curvature in the lepton and quark spaces which are given, respectively, by

$$
I_l = \mathrm{Tr}(C_l(\theta_\rho + \theta_\eta)^2 D_l^{-4})
$$

$$
I_q = \mathrm{Tr}(C_q(\theta_\rho + \theta_\eta)^2 D_q^{-4})
$$

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where $\theta = d\rho + \rho^2$ is the curvature of ρ, and C_l, C_q are constant elements of the algebra. After projecting out the auxiliary fields, it is possible to show that the above bosonic action reproduces all the bosonic interactions of the standard model, and with the same number of parameters [2,4]. The gauge fields and Higgs fields are unified in this formalism. Only when C_l and C_q belong to the center of the algebra one gets fixed values for the top quark mass and Higgs mass, but these relations cannot be maintained at the quantum level [8].

The main advantage of following the non-commutative construction of the standard model is that one gets a geometrical significance for the Higgs field as well as a prediction for the nature of the Higgs sector (in this case one doublet). The distance between the two copies of the four-manifold is given by the inverse of the weak scale. At very low energies the two copies are superimposed on each other, and only at the energy scale of 100 Gev one can "observe" the splitting of the two copies through the Higgs interactions.

4. Unification models

It is natural to ask whether one can go beyond the standard model in noncommutative geometry. In particle physics it is popular to consider larger groups such as $SU(5)$ and $SO(10)$ [9] which contain $SU(3) \times SU(2) \times U(1)$. The main advantage is that the fermion fields could be unified in one or two group representations. The nicest example of this is the 16_s spinor representation of $SO(10)$ which contains all known particles and in addition a right-handed neutrino that can be used to give a very small Majorano mass to the left-handed neutrino. The gain in simplicity does not make the theory more predictive. This is mainly due to the arbitrariness in the Higgs sector. In unified models there are many possible Higgs representations that can do the symmetry breaking of the gauge group down to $SU(3) \times U(1)$ in a satisfactory way. If the Higgs sector and the breaking mechanism could be uniquely determined then grand unified theories could become more predictive.

As we have seen in the non-commutative construction of the standard model, the Higgs sector was completely determined, and one hopes that this feature will continue to be present in unified theories. We have also associated the distance between the two copies of the manifold with the energy scale of the electroweak breaking. For unified theories the breaking is done at least at two different scales and for this one must require the discrete group to contain more than two points [6]. Another starting point is the grouping of the spinor fields and the Hilbert space associated with the spinors.

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For example, if we arrange the leptons in the form $Q = \begin{pmatrix} q_L \\ q_R \end{pmatrix}$ where $q = \begin{pmatrix} \nu \\ e \end{pmatrix}$, then the corresponding algebra $\mathcal{A}_2$ will be $M_2(C) \oplus M_2(C)$. The simplest and most natural possibility corresponds to a discrete space of four points, and where the fermions are arranged in the form $Q = \begin{pmatrix} q_L \\ q_R \\ q_L^c \\ q_R^c \end{pmatrix}$, and the representation π acting on $\mathcal{A}$ is given by $\pi(a) = \mathrm{diag}(a_1, a_2, \overline{a_1}, \overline{a_2})$ where a_1 and a_2 are two by two matrices. One can convince himself that the resulting model would be the $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ with the Higgs fields in the representations $(2,2), (3,1) + (1,3)$ of $SU(2)_L \times SU(2)_R$. In case the fermionic representations are not simple such as the $\overline{5} + 10$ for $SU(5)$, then the situation becomes complicated because one must specify the action of elements of the algebra on the spinors in ways which does not seem to be very natural. In this respect it is preferable to take fundamental, or spinor representations of the algebra for the space-time spinors. We can therefore summarize the steps needed in constructing a non-commutative model.

1. Specify the representation of the fermions.

2. Choose the number of discrete points and the symmetry between them.

3. Deduce the appropriate algebra and the map π acting on the Hilbert space of spinors.

4. Write down the Dirac operator acting on elements of the algebra, and in particular choose the mass matrices to correspond to a good vacuum of the Higgs fields.

The above rules will completely fix the fermionic and bosonic sectors of the model. We note that only certain vacuum expectation values (vevs) of the Higgs fields are allowed because the potential is orthogonal to the auxiliary space, and for a generic vev the auxiliary space is as big as the algebra itself. To give a concrete example we take the chiral space-time spinors $P_+\psi$ to be in the 16_s representation of $SO(10)$, where P_+ is the $SO(10)$ chirality operator, and the number of discrete points to be three [10]. The Hilbert space is taken to be that of the spinor $\Psi = \begin{pmatrix} P_+\psi \\ P_+\psi \\ P_-\psi^c \end{pmatrix}$ and where $\psi^c = BC\overline{\psi}^T$, C being the charge conjugation matrix and B the $SO(10)$ conjugation matrix. This ordering will guarantee that the spinors can acquire masses. The algebra $\mathcal{A}_2$ is takent to be $P_+\Big(\mathrm{Cliff}\,SO(10)\Big)P_+$. Let π_0 denote the representation of $\mathcal{A}$ on the Hilbert space $h_1 \otimes \tilde{h}_2$ on square-integrable spinors for SO(1,3) $\times$ SO(10), where $\tilde{h}_2 = \mathbb{C}^{32}$ is the 32-dimensional vector space on which $\mathcal{A}_2$ acts. Let $\bar{\pi}_0$ denote

the anti-representation defined by $\bar{\pi}_0(a) = B\,\overline{\pi_0(a)}\,B^{-1}$. We define $\pi(a)$ by setting $\pi(a) = \pi_0(a) \oplus \pi_0(a) \oplus \bar{\pi}_0(a)$. We choose the Dirac operator D to be given by

$$D = \begin{pmatrix} \partial\!\!\!/ \otimes 1 \otimes 1 & \gamma_5 \otimes M_{12} \otimes K_{12} & \gamma_5 \otimes M_{13} \otimes K_{13} \\ \gamma_5 \otimes M_{21} \otimes K_{21} & \partial\!\!\!/ \otimes 1 \otimes 1 & \gamma_5 \otimes M_{23} \\ \gamma_5 \otimes M_{31} \otimes K_{31} & \gamma_5 \otimes M_{32} \otimes K_{32} & \partial\!\!\!/ \otimes 1 \otimes 1 \end{pmatrix}$$

where the K_{mn} are 3×3 family-mixing matrices commuting with $\pi(\mathcal{A})$. We impose the symmetries $M_{12} = M_{21} = \mathcal{M}_0$, $M_{13} = M_{23} = \mathcal{N}_0$, $M_{31} = M_{32} = \mathcal{N}_0^*$, with $\mathcal{M}_0 = \mathcal{M}_0^*$. Similar conditions are imposed on the matrices K_{mn}. For D to leave the subspace h invariant, $\mathcal{M}_0$ and $\mathcal{N}_0$ must have the form

$$\mathcal{M}_0 = P_+\big(m_0 + im_0^{IJ}\Gamma_{IJ} + m_0^{IJKL}\Gamma_{IJKL}\big)P_+$$
$$\mathcal{N}_0 = P_+\big(n_0^I\Gamma_I + n_0^{IJK}\Gamma_{IJK} + n_0^{IJKLM}\Gamma_{IJKLM}\big)P_-$$

where $\Gamma_{I_1 I_2 \cdots I_n} = \frac{1}{n!}\Gamma_{[I_1}\Gamma_{I_2}\cdots\Gamma_{I_n]}$ are antisymmetrized products of the gamma matrices.

Next we define an involutive "representation" $\pi : \Omega^*(\mathcal{A}) \Rightarrow B(h)$ of $\Omega^*(\mathcal{A})$ by bounded operators on h; ($B(h)$ is the algebra of bounded operators on h). It is straightforward to compute $\pi(\rho)$ and one gets [10]

$$\pi(\rho) = \begin{pmatrix} A & \gamma_5\mathcal{M}K_{12} & \gamma_5\mathcal{N}K_{13} \\ \gamma_5\mathcal{M}K_{12} & A & \gamma_5\mathcal{N}K_{23} \\ \gamma_5\mathcal{N}^*K_{31} & \gamma_5\mathcal{N}^*K_{32} & B\overline{A}B^{-1} \end{pmatrix}$$

where the fields A, $\mathcal{M}$ and $\mathcal{N}$ are given in terms of the a^i and b^i by

$$A = P_+\big(\sum_i a^i\partial\!\!\!/ b^i\big)P_+$$
$$\mathcal{M} + \mathcal{M}_0 = P_+\big(\sum_i a^i\mathcal{M}_0 b^i\big)P_+$$
$$\mathcal{N} + \mathcal{N}_0 = P_+\big(\sum_i a^i\mathcal{N}_0 B\overline{b^i}B^{-1}\big)P_-$$

We can expand these fields in terms of the $SO(10)$ Clifford algebra. The self-adjointness condition on $\pi(\rho)$ implies, after using the hermiticity of the Γ_I matrices, that all the fields appearing in the expansion of $A, \mathcal{M}$ are real, because both are self-adjoint, while those in $\mathcal{N}$ are complex. Equating the action of A on ψ and ψ^c reduces it to an $SO(10)$ gauge field. The structure of the Higgs fields is completely determined to be given by $16_s \times 16_s$ and $16_s \times \overline{16}_s$. Specifying $\mathcal{M}_0$ and $\mathcal{N}_0$ determines the breaking pattern of $SO(10)$. For the potential to give these vevs as its

minimum the auxilary space should be smaller than the algebra. This imposes severe conditions on the choices of $\mathcal{M}_0$ and $\mathcal{N}_0$. If the number of discrete points in this model is changed, this would not change the nature of the Higgs fields, but only the coefficients of their couplings. The only modification we can make is to add to the spinor ψ a singlet spinor so as to give the right-handed neutrino a Dirac mass. In this case a Higgs field 16_s will also be present. We deduce that the most important advantage of the non-commutative construction is the prediction of the Higgs representations once the spinors of the model are specified. This gives well defined models which can be analyzed in detail. But there is no reduction in the number of parameters corresponding to fixed Higgs representation and an admisable breaking pattern. From this discusion one can deduce that only a very small number of models can be constructed and for each model the Higgs representation is fixed, and the symmetry breaking pattern severely restricted. It will be very interesting to classify these models, and to study the allowed symmetry breakings, and to investigate the phenomenology of the promissing cases.

5. Supersymmetry in non-commutative geometry

Theories with space-time supersymmetry has many nice properties which are well known [11]. It is then tempting to construct non-commutative actions whose classical part has space-time supersymmetry. The simplest example is provided by the $N = 1$ super Yang-Mills theory in four dimensions . The action is given by [12]:

$$I = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{2} \overline{\lambda^a} \gamma^\mu D_\mu \lambda^a \right),$$

where λ^a is a Majorana spinor in the adjoint representation of a gauge group G, $F_{\mu\nu}^a$ is the field strength of the gauge field A_μ^a and D_μ is a gauge covariant derivative. This action is invariant under the supersymmetry transformations

$$\delta\lambda^a = -\frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}^a \epsilon,$$
$$\delta A_\mu^a = \overline{\epsilon} \gamma_\mu \lambda^a,$$

To reformulate this action using the methods of non-commutative geometry [1], we first define the triple $(\mathcal{A}, h, D)$ where h is the Hilbert space $L^2(M, \tau, \sqrt{g}d^4x) \otimes C^n$ of spinors on a four-dimensional spin manifold M, $\mathcal{A}$ is the involutive algebra $\mathcal{A} = C^\infty(M) \otimes M_n(C)$ of $n \times n$ matrix valued functions, and D the Dirac operator $D = \partial\!\!\!/ \otimes 1_n$ on h. The free part of the fermionic action is written as $\frac{1}{2}(\lambda, [\partial\!\!\!/, \lambda])$, where $(,)$

denotes the scalar product on $L^2(S, \tau, \sqrt{g}d^4x)$ given by

$$(\psi_1, \psi_2) = \int_M \sqrt{g}d^4x\tau\langle\psi_1(x), \psi_2(x)\rangle,$$

where τ is a normalised trace on $\mathcal{A}$, and $\langle,\rangle$ denotes the hermitian structure on the left module $\mathcal{E}$ which will be taken to be equal to $\mathcal{A}$. Let ρ be a self-adjoint element in the space $\Omega^1(\mathcal{A})$ of one-forms: $\rho = \sum_i a^i db^i$, Then $\pi(\rho) = \sum a[D, b]$ is equal to $\gamma^\mu A_\mu$ where $A_\mu = \sum a\partial_\mu b$. Since ρ is self-adjoint and γ^μ is antihermitian, then $A_\mu^* = -A_\mu$. The curvature of ρ is $\theta = d\rho + \rho^2$ where $\theta \in \Omega^2(\mathcal{A})$. A simple calculation shows that $\pi(d\rho) = \gamma^{\mu\nu}\partial_\mu A_\nu + \sum \partial^\mu a\partial_\mu b$. If $\pi(\rho) \in \text{Ker}(\pi)$, then $\pi(d\rho) = \sum \partial^\mu a\partial_\mu b = -\sum a\partial^\mu\partial_\mu b$, is an independent scalar function. The choice of $\pi(d\rho)$ in $\pi(\Omega^2(\mathcal{A})) \setminus \pi(d\text{Ker}\pi_{\setminus\Omega^1(\mathcal{A})})$ is uniquely determined to be orthogonal to all auxiliary fields, with respect to the inner product on $\Omega^2(\mathcal{A})$. From this we deduce that, modulo the auxiliary field (i.e. the kernel of $\pi(d\rho)$), $\pi(\theta) = \gamma^{\mu\nu}F_{\mu\nu}$. The Yang-Mills action is

$$\frac{1}{2}\text{Tr}_w(\theta^2 D^{-4}) = \int \sqrt{g}d^4x\text{Tr}(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}),$$

where Tr_w is the Dixmier trace [1]. The interacting fermionic action is

$$\frac{1}{2}(\lambda, [D + \rho, \lambda]) = \frac{1}{2}\int \sqrt{g}d^4x\text{Tr}(\overline{\lambda}\gamma^\mu[\partial_\mu + A_\mu, \lambda]).$$

The supersymmetry transformation for λ and ρ take the simple form

$$\delta\lambda = -\pi(\theta)\epsilon,$$

$$\delta\pi(\rho) = \overline{\epsilon}E_a\lambda(E_a),$$

where E_a is a local orthonormal basis of $\Omega_D^1(\mathcal{A}) \equiv \Omega^1(\mathcal{A}) \setminus (\text{Ker}\pi + d\text{Ker}\pi)$. In our case the basis is $E^a = \gamma^a$.

We next consider the $N = 2$ super Yang-Mills action [13]. It is given by

$$I = \int d^4x\Big(-\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{2}D_\mu S^a D^\mu S^a + \frac{1}{2}D_\mu P^a D^\mu P^a + \overline{\chi}^a\gamma^\mu D_\mu\chi^a$$

$$- if^{abc}\overline{\chi}^a(S^b - i\gamma_5 P^b)\chi^c - \frac{1}{2}\big(f^{abc}S^b P^c\big)^2\Big),$$

where S^a and P^a are a scalar and pseudoscalar fields, and χ^a is a Dirac spinor, all in the adjoint representation of the gauge group. This action is invariant under the transformations:

$$\delta A_\mu^a = \overline{\epsilon}\gamma_\mu\chi^a - \overline{\chi^a}\gamma_\mu\epsilon,$$

$$\delta P^a = \overline{\chi^a}\gamma_5\epsilon - \overline{\epsilon}\gamma_5\chi^a,$$

$$\delta S^a = i(\overline{\chi^a}\epsilon - \overline{\epsilon}\chi^a),$$

$$\delta\chi^a = \big(-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}^a - \gamma_5 f^{abc}P^b S^c + i\gamma^\mu(D_\mu S^a - i\gamma_5 D_\mu P^a)\big)\epsilon.$$

From our experience with the non-commutative construction of the standard model, and since the action contains a complex scalar field unified with a gauge field, an obvious guess is to take the non-commutative space to be $M_4 \times$ (two points), with the algebra

$$\mathcal{A} = C^\infty(M_4) \otimes M_n(C) \oplus C^\infty(M_4) \otimes M_n(C),$$

and the Dirac operator

$$D = \begin{pmatrix} \slashed{\partial} \otimes 1_n & i\gamma_5 \otimes \phi_0 \\ -i\gamma_5 \otimes \phi_0^* & \slashed{\partial} \otimes 1_n \end{pmatrix},$$

acting on the Hilbert space of spinors of the form $\lambda = \begin{pmatrix} L\chi \\ R\chi \end{pmatrix}$, where $L = \frac{1}{2}(1 + \gamma_5)$ and $R = \frac{1}{2}(1 - \gamma_5)$, and χ is a Dirac spinor. Elements of $\mathcal{A}$ are taken to be operators of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ where a is a smooth function on M_4 with values in $M_n(C)$. The parameters ϕ_0 are taken to be arbitrary except for the constraint $[\phi_0, \phi_0^*] = 0$. A self-adjoint element ρ in the space $\Omega^1(\mathcal{A})$ has the representation

$$\pi(\rho) = \begin{pmatrix} \gamma^\mu A_\mu & i\gamma_5 \phi \\ -i\gamma_5 \phi^* & \gamma^\mu A_\mu \end{pmatrix},$$

where $A_\mu = \sum a\partial_\mu b$, $\phi + \phi_0 = \sum a\phi_0 b$ and $\phi^* + \phi_0^* = \sum a\phi_0^* b$. The fermionic action can now be simply written as

$$\frac{1}{2}\big(\lambda, [D + \pi(\rho), \lambda]\big) = \frac{1}{2}\int \sqrt{g}d^4x \mathrm{Tr}\big(\overline{\lambda}[D + \pi(\rho), \lambda]\big).$$

A straightforward calculation [14] shows that the bosonic part of the noncommutative action is given by

$$\frac{1}{4}\mathrm{Tr}_w\big(\theta^2 D^{-4}\big) = \int \sqrt{g}d^4x \mathrm{Tr}\big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi D^\mu\phi + \frac{1}{8}([\phi, \phi^*])^2\big).$$

Continuing from Euclidean to Minkowski space and inserting $\phi = S - iP$, we exactly recover the bosonic part of the supersymmetric action . The supersymmetry transformations are now very simple:

$$\delta\lambda = -\pi(\theta)\epsilon$$

$$\delta\pi(\rho) = \big(\overline{\epsilon}E_i\lambda - \overline{\lambda}E_i\epsilon\big)E_i,$$

where E_i a local orthonormal basis of $\Omega_1(\mathcal{A})$, and ϵ has the same representation as λ. In this case the basis can be taken to be $E_a = \gamma_a \otimes 1_2$, $E_5 = i\gamma_5 \otimes \tau_1$, $E_6 = i\gamma_5 \otimes \tau_2$, where τ_1 and τ_2 are Pauli matrices.

The same analysis can be repeated for the $N = 4$ super Yang-Mills action [15], and one can show that it admits a non-commutative construction [14]. When this idea is attempted with $N = 1$ super Yang-Mills coupled to matter, one finds that this is possible, after some assumptions are made, provided no general superpotential is taken. The interesting problem to solve is to find the special kind of superpotentials compatible with non-commutative geometry.

6. Gravity in non-commutative geometry

One of the original motivations for seeking a new structure of space-time was to include gravity with the other interactions in a consistent way. Therefore the question of finding the gravitational action in non-commutative geometry must be posed. One can make the construction based on generalizing the basic notions of Riemannian geometry [16]. For this one defines the metric as an inner product on cotangent space. One shows that every K cycle over $\mathcal{A}$ yields a notion of cotangent bundle associated with $\mathcal{A}$ and a Riemannian metric on the cotangent bundle $\Omega_D^1(\mathcal{A})$. With the connection ∇ we define on $\Omega_D^1(\mathcal{A})$ the Riemann curvature of ∇ by $R(\nabla) :=$ $-\nabla^2$, and the torsion $T(\nabla) = d - m.\nabla$, where m is the tensor product operator. Requiring the connection to be unitary and the torsion to vanish we obtain a Levi-Civita connection. If $\Omega_D^1(\mathcal{A})$ is a free, finitely generated module then it admits a basis $e^A, A = 1, 2, \ldots N$, and the connection $\omega^A_B \in \Omega_D^1(\mathcal{A})$ is defined by $\nabla e^A =$ $-\omega^A_B \otimes_{\mathcal{A}} e^B$. Let $T^A \in \Omega_D^2(\mathcal{A})$ be the components of the torsion $T(\nabla)$ defined by $T^A = T(\nabla)e^A$. Then

$$T^A = de^A + \omega^A_B e^B$$

Similarly we define $R^A_B \in \Omega_D^2(\mathcal{A})$ by $R(\nabla)e^A = R^A_B \otimes_{\mathcal{A}} e^B$. Then

$$R^A_B = d\omega^A_B + \omega^A_C \omega^C_B.$$

The analogue of the Einstein-Hilbert action is

$$I(\nabla) := \kappa^{-2} < R^A_B e^B, e_A > +\Lambda < 1, 1 >$$
$$= \kappa^{-2} \int_M \mathrm{tr}(R^A_B e^B (e_A)^*) + \Lambda \int 1,$$

where κ^{-1} is the Planck scale. When this formalism is applied to the product $M_4 \times Z_2$ one finds that [16]

$$I(\nabla) = 2 \int_M (\kappa^{-2} r - 2\partial_\mu \sigma \partial^\mu \sigma + \Lambda)\sqrt{g} d^4 x$$

where r is the scalar curvature of the classical Levi-Civita connection, and σ is a massless scalar field couples to the metric of M. To better understand the role of the field σ we can study the coupling of gravity to the Yang-Mills sector [17]. In the case of the standard model the field $\phi = e^{-\kappa\sigma}$ replaces the electroweak scale. In other words, the vev of the field ϕ determines the electroweak scale. To determine the σ dependence in the Yang-Mills action of the standard model, we consider the σ dependence in the Dirac operator. For example, the leptonic Dirac operator is

$$D_l = \begin{pmatrix} \gamma^a e_a^\mu(\partial_\mu + \ldots) \otimes 1_2 \otimes 1_3 & \gamma_5 e^{-\kappa\sigma} \otimes M_{12} \otimes k \\ \gamma_5 e^{-\kappa\sigma} \otimes M_{12}^* \otimes k^* & \gamma^a e_a^\mu(\partial_\mu + \ldots) \otimes 1_3 \end{pmatrix}$$

The bosonic action is invariant under rescaling of the Dirac operator $D \to e^{-w}D$, as this implies $g_{\mu\nu} \to e^{2w}g_{\mu\nu}$ and $\kappa\sigma \to \kappa\sigma + w$. The quantum corrections to the classical potential will depend on σ, and the vev of σ could be determined from the minimization equations. This excercise, when applied to the standard model, gives one extra equation. The minimization equations could only be solved with a heavy top quark mass in the region $104 \leq m_t \leq 147$ Gev , and a Higgs mass $m_H = 1200$ Gev . But this lies in the region where perturbative methods fail. Therefore at present one cannot make a prediction for a signature of non-commutative geometry when applied to the standard model.

7. Conclusions and comments

We have seen that non-commutative geometry provides a powerful tool to deal with spaces that could not be dealt with using the usual methods of differential geometry. The examples that we have considered so far are based on the simple geometry of a manifold times a discrete set of points, and it appears that to make more progress we have to consider space-times which are completely non-commutative. Since all the data of a non-commutative space are encrypted in the triplet $(\mathcal{A}, h, D)$, one would like to determine the characteristics of a physically good set. Another important point to study is the search for the symmetries present in a non-commutative action, and how to quantize such actions, and whether the new symmetries, if present, could be maintained at the quantum level. Another important problem is to understand better the connection between space-time and non-commutative geometry, and in particular, to find out the special proporties of Dirac operators of supersymmetric theories. The study of gravitational fields for a non-commutative spaces has to be studied further, and in particular the dynamical degrees of freedom present in such theories. Therefore, although the methods of non-commutative geometry when ap-

plied to simple spaces give rise to very nice models in particle physics, many open problems remain. We hope that the solution to some of these problems would shed light on the structure of the unified theory, and make further progress possible.

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